

# A classical fermionic system that follows the fundamental rules of Quantum Mechanics.

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## Abstract

It is feasible to obtain any basic rule of the already known Quantum Mechanics applying the Hamilton-Jacobi formalism to an interacting system of 2 fermionic degrees of freedom. The interaction between those fermionic variables unveils also a primitive spin and *zitterbewegung*.

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Since the beginning of the past century, the Physicists have witnessed the born of a new branch of the knowledge, known as Quantum Mechanics [1]-[4]. The study of macroscopic and microscopic phenomena suggested the application of new theories and models, because did not follow the classical Physics, supposed to be finished at that time, in the sense that was able to explain, apparently, any kind of event in the nature. The black body radiation had to be understood introducing a new fundamental constant, named Planck's, which value  $h \sim 10^{-34}$  Js, represents twice unusual considerations: The dimension of the particles involved in the phenomena and the statistical behavior of it. After that, the deeper study of the matter, the atom, showed to the scientists, that a set of new concepts and rules needed to be created in order to explain its properties, specifically the Hydrogen, the simplest one. Then it was proposed an equation to predict the behavior of the nature at the microscopic world. Using that equation of Schördinger it was feasible to explain the dynamics of electrons and matter, in a way quite close to the observed experimentally, during the starting years of study. However, the unknown previously second order partial differential equation by a wave function for the electron, was the beginning of a quite famous confrontation in the scientific community, because that one represented something not conceivable so easily, a wave function of matter. Even more, that function had probabilistic properties, indicating indeterminism at the time of describing the movement of particles, something unusual at those years, were the Classical Dynamics had been used to understand -apparently- all the surrounding phenomena [5]. According to the information at hand, never has been understood the role and the intrinsic meaning of that wave function since neither the physical nor the philosophical point of view. At the scientific literature it is discussed the consequences of Quantum Mechanics, and its interpretation in order to understand deterministically that branch of the Physics, cause and effect. In spite of those efforts, like Bohmian Mechanics [6], that equation and the concepts surrounding it, are still a cornerstone of knowledge when the people tries to comprehend something that is not directly observable by our eyes. Very recently, it has been published an article where it is studied a hypothetical system, consisting of just twice fermionic degrees of freedom (DOF). The aim of that is to handle odd Grassmann variables at the classical level, waiting to obtain some clues of the missed determinism at the quantum world, and the anticommutative algebraic properties of the operator representing the intrinsic property of the electron, the spin. That is unusual also in the common Science, because it is based on even Grassmann or commutative variables.

It is not so easy to accept an anticommutative nature, and also a mathematics that is not usually thought at any degree of a traditional education. Three decades ago the Lagrangian formalism for bosonic and fermionic DOF has been developed, including an acceptable description of the spin [7]. However, there are still attempts to avoid the odd Grassmann DOF because of the difficulty in handling algebraically those variables [8]-[9], and because there are still several doubts concerning to the formulas and principles of Quantum Mechanics, that are of our general interest to have in mind clearer, like the spin itself [10]-[11]. In this letter, it shown a new way of understanding some of the main principles and rules of Quantum Mechanics, following the consequences of solving an interacting fermionic system with the classical Hamilton-Jacobi formalism [12]. That problem seems to be solved for a suppersymmetric Lagrangian [13] but considering a slightly different procedure than the described at [12]; the frontier to Quantum Mechanics is not mentioned there. The procedure for quantizing a classical system isn't still fully justified since the mathematical and physical points of view, even more if it is relativistic and/or fermionic [14]-[15]. That's why it is important to clarify that ambiguity in knowledge obtaining the rules of Quantum Mechanics since still most basic principles. The solution according to [12] let us to realize of the next already know rules since the scheme of fermionic DOF: (1) The equation of Schördinger, (2) The brackets of Poisson, (3) The principle for exclusion of Pauli, (4) The II postulate of Bohr for quantizing the energy, (5) The rule for quantizing of Bohr-Sommerfeld, (6) The principle of uncertainty by Heisenberg, (7) The wavelength of De Broglie/Compton, (8) The spin, and (9) The equation of Dirac. All of them, in agreement to the previously known rules of Quantum Mechanics, corresponding to bosonic DOF.

**1. The equation of Schördinger.** A fermionic system [12] described by the Lagrangian  $L = i(\psi_1\dot{\psi}_2 + \psi_2\dot{\psi}_1) + k\psi_1\psi_2$  and the constrains  $\phi_i = \pi_i + i\psi_i \approx 0$ ,  $i = 1, 2$ , is feasible to be studied after applying the Hamilton-Jacobi formalism if  $\psi_2 = \psi_1^*$  and  $[\psi_1, \psi_2]_+ = 0$ . The solution for that system is  $\psi_1(t) = \xi_1 \exp(ik/2)t$ , where  $\xi = -a \exp(-ic)\rho_2$ , in which  $a, c$  and  $\rho_2$ , are arbitrary even and odd Grassmann constants, respectively. As it has been shown at that reference, it is feasible to write the solution as a vector such as  $\Psi = \Psi_o \exp(\alpha S)$ , where  $S = s_1\rho_1\psi_1 + s_1\rho_1\psi_2 + (s_{3o} + s_3\rho_1\rho_2)\psi_1\psi_2$  is the Action,  $s_1 = a^*(s_{3o} + i) \exp[i(k/2)t]$ ,  $\alpha = s_3/|a|^2 \in \mathbb{R}$ , and  $\Psi = (\psi_1, \psi_2)$ . That way of expressing the solution is reminiscent of the corresponding to the wave equation of Electrodynamics  $\Psi = \Psi_o \exp(iS/\hbar)$ , useful to derive since here, the Schördinger equation after applying the WKB approximation, and

“assuming” that  $E = h\nu$ . In this case it is not necessary to do that, but just looking for the equation corresponding to the fermionic solution, so

$$i\tilde{\hbar}\frac{\partial\Psi}{\partial t} = \left(\frac{\tilde{\hbar}k}{2}\sigma_3 + ik\psi_1\psi_2\right)\Psi, \quad (1)$$

being  $\sigma_3$  the 3rd Pauli matrix, and  $\tilde{\hbar} \equiv \alpha^{-1}$ , a reduced constant of Planck. This equation is a simile for the Schrödinger's, that is feasible to be modified in order to include bosonic DOF. In spite of its mystery, the equation of Schrödinger can be deduced also, following another type of mathematical arguments, that not necessarily has to do with odd Grassman variables. [16, 17] Note that the vectorial function  $\Psi$  has mathematical properties, that follow the algebra of Grassmann [18] and that it has nothing to do to probability *per se*. To deduce the next quantum set of rules, it is not required to explain the deeper physical meaning of  $\Psi$ .

**2. The brackets of Poisson and commutator.** The solution to the model for 2 fermionic variables can be used to calculate their classical Poisson brackets, obtaining that  $\{\psi_i, \pi_j\} = -\delta_{ij}$ , meanwhile

$$[\tilde{\psi}_i, \tilde{\pi}_j] = i\tilde{\hbar}\delta_{ij} \quad (2)$$

where  $\tilde{\psi}_i = \psi_i/\sqrt{2s_3}$  and  $\tilde{\delta}_{ij} \equiv \rho_i\rho_j$ , showing supersymmetry to the already known bosonic variables, due to the commutator  $[q_i, p_j] = i\hbar\delta_{ij}$  that suggests that  $\tilde{\hbar} \equiv \hbar$  [7]-[12]. The canonical behave of the commutator is another result that helps us to trust in the Hamilton-Jacobi formalism already applied, for the specific fermionic system selected. Hence, the procedure [12] is not only justified by the previous result, but by others [19] that show symmetry in the brackets of Poisson for odd and even Grassmann DOF.

**3. The principle for exclusion of Pauli.** Accordingly to this, it is not possible for two electrons to own the same quantum numbers or to occupy the same state. That physical property is represented mathematically by the non commutativity of the variables representing to each degree of freedom of the spin. In our case, that algebraic property of the odd Grassman variables already selected, can be represented in the bidimensional complex coordinate plane, selecting a DOF such as  $\psi_1 = \psi_{1x} + i\psi_{1y}$ , and as a consequence, the second one being  $\psi_1^* = \sigma_3\psi_2$ , that is just half turn at that complex plane. As a consequence, the non commutativity of the fermionic DOF turns out to twice different states that are related like spin's. Similarly to Ref. [20], it is required here an interaction, at least at the fermionic

order, to find a physical property, like it is the spin projected at the  $z$  axe. Reference [20] discusses the case of a hidden supersymmetry at 3D, that appears when it is assumed a coupling spin-orbit. By the inverse situation [12], the interaction of the 2 fermionic DOF doesn't need that.

**4. The II postulate of Bohr for quantizing the energy.** The eigenvalues of Eq. (1) gives us a shift in energy

$$\Delta E = \tilde{\hbar}k \quad (3)$$

that is reminiscent of the II classical postulate of Bohr for quantizing the energy  $\Delta E = \hbar\omega$ , being  $\omega$  or  $k$  the angular frequency of radiation. For this rule, there isn't also an evident dynamical scenario, since it be easily interpreted the meaning of the energy corresponding to the states of the fermionic system. In other words, in spite that the model is fully based on 2 odd Grassmann DOF, there is an intrinsic dynamics that owns 2 values of energy, for any of the complex conjugated states. That primitive interaction between fermionic variables is also undeductable from an even Grassmann model, like it happens in Quantum Mechanics. The single interacting term at the Hamiltonian  $H = -k\psi_1\psi_2$  in Eq. (1), hasn't direct contribution to energy, from the fact that it vanishes at the time of multiplying by the vector  $\Psi$ . Then, the contribution to energy is only coming from the therm  $\propto \sigma_3$ .

**5. The rule to quantize of Bohr-Sommerfeld.** Here, the typical integral of action variables is calculated straightforwardly, according to [7]

$$J = \oint \pi_i d\psi_i = \tilde{\hbar}k\tilde{\delta}_{12}, \quad i = 1, 2 \quad (4)$$

where it has been assumed that  $s_3 \equiv \nu$ , the frequency. As a consequence, after  $n$  cycles:  $E_n = n \cdot \tilde{\hbar}k$ , which corresponds to the well known result  $E_n = n \cdot \hbar\omega$ . The physical meaning of this rule could be clearer if it is referred to a complex plane, being a simile for the classical, by even Grassman variables.

**6. The principle of uncertainty by Heisenberg.** It is straightforward to obtain this, since the solution for the fermionic system

$$\Delta\psi_i \cdot \Delta\pi_i \sim \tilde{\hbar}, \quad i = 1, 2 \quad (5)$$

that is of the same type to the corresponding for bosonic DOF. None statistic had to be applied in order to obtain the previously uncertain result. The Eq. (5) is meaningful because

there is an intrinsic ambiguity at the time of setting simultaneously the value of the fermionic variables  $\psi$  and its corresponding momentum  $\pi$ , like it happens at Quantum Mechanics.

**7. The wavelength of De Broglie/Compton.** After this steps, it is possible to search what kind of physical variables could own  $a$  if  $[\tilde{h}] = [\hbar]$ . Starting by the fact of the squaring of  $a$ , if  $|a| \equiv \sqrt{[m] \cdot [c]^2}$  and  $[s_3] \equiv [\nu]$ , then  $[\tilde{h}] = [mc][\lambda]$ , then it is possible to deduct that

$$\lambda = \frac{\tilde{h}}{mc} \quad (6)$$

because if  $\tilde{h} = \hbar$ , then  $\lambda_c = \hbar/m_e c$ , is the wavelength of matter corresponding to the movement of a free electron (Compton's). At the situation in interest, there isn't a movement as it is known in Classical Dynamics, however, it seems that there is a wave corresponding to the intrinsic fermionic property of the system.

**8. The spin.** The Hamilton-Jacobi procedure for a system of 2 fermionic DOF, let us to discovery an operator at Eq. (1) that looks the same than the already known for the electron's spin  $S = (\hbar/2)\sigma_3$ , by writing  $\tilde{h} = \hbar$ . As a consequence, even this supposed to be an intrinsic and non classical quantity, is expressed as a function of classical, deterministic variables. In other words, the spin turns out to be a property understandable since the mathematical point of view, asking only by the non commutativity of 2 fermionic, odd Grassmann variables that has to be complex conjugated between them. Physically, it is a consequence of the interaction between fermionic variables (see Eq. (2)).

**9. The equation of Dirac.** Finally, in order to explore the frontier between unrelativistic and relativistic Quantum Mechanics, just set the fermionic variables and their corresponding momenta into a single vector of 4 elements, or bi-spinor  $\tilde{\Psi} = \gamma_o \tilde{\Psi}^o \exp(\alpha S)$ , where  $\gamma_o$  is one of the matrices of Dirac [2]. It corresponds to apply the equation of Dirac in its primitive version as follows, calculating the eigenvalues of the temporal Hamiltonian  $H_o = \gamma_o mc^2$  on the bi-spinor  $\tilde{\Psi}$ . It is obtained that

$$\tilde{h}k = \pm 2mc^2, \quad (7)$$

a relation that is consistent with the guessing of dimensions previously done at Eq. (7). As it can be seen, considering just the temporal contribution to the equation of Dirac, it is found a constrain between the "nonrelativistic"  $\tilde{h}k$  and the "relativistic"  $2mc^2$  energies. A situation already predicted at Eq. (6), because of the algebraic expression for  $\tilde{h}$ , as it is defined at Eq. (1), and already obtained by Dirac [2, 3].

As it can be observed rule *per* rule, it is not necessary to do a non classical suggestion in order to find the quantum relations already known for bosonic DOF in Quantum Mechanics, which are obtained completely after setting  $\psi_i \rightarrow q_i$ ,  $\pi_i \rightarrow p_i$ , and  $\tilde{\hbar} = \hbar$ . An advantage of the analysis previously done, is that the fermionic system is quantic *per se*, but that  $\tilde{\hbar}$  could be almost any parameter having a dimension of angular momentum, and which value is not necessarily  $\hbar$ . The procedure and results shown here are not limited to the already known rules of Quantum Mechanics. It is not difficult to think on a system where the generalized constant of Planck, be expressed in such a way that be permissible to apply the fermionic model, without dimensional contradictions. For instance, a physical situation such that  $\tilde{\hbar}$  be written as  $m_c \cdot c^2 \cdot t_c$ , where  $m_c, t_c$  are unknown mass and time, playing a role of like the units of Planck, that are associated to a supposed unification of the fundamental forces in nature [21]. The shift in energy at Eq. (7), due to the term of interaction  $H = -k\psi_1\psi_2$  has appeared previously during several attempts to explain a physical phenomena related to an oscillatory movement of the electron, named in German *zitterbewegung* [3], done at a frequency  $\nu \sim 10^{21}$  Hz. The electron is supposed to move freely at almost the speed of light. Some authors attribute that vibration of the electron to a relativistic effect, while others consider that the term is just a mathematical artifact [22]-[26]. The results described here, show a contribution to the energy, coming from the interaction between fermionic variables, that is feasible associable to the relativistic energy. It is true, that using odd Grassmann variables appear a contribution to the energy, coming just from that interaction in states of positive and negative energies [3], and that it is a simile for the already known at the “quantum” level. Apparently, it is necessary to equate that term of energy to the relativistic one through the equation of Dirac. However, it has been already demonstrated at Eq. (7), that selecting properly the values of the coefficients  $a$  and  $s_3$ , it is feasible to reproduce that oldy result, without modifying the dynamics chosen *a priori*, letting that the contribution to the energy be an effect of the underlying non commutative algebra, and due to the interaction between the 2 complex conjugated fermionic DOF. It is worthy the research that has been done in order to clarify the *zitterbewegung* since its discovery by Schördinger, but it is still a phenomena that has not been fully described at the time of

being considered its dynamics in detail.

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